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ON ONE CLASS OF INVERSE PROBLEMS OF VARIATION IN SHAPE OF VISCOELASTIC PLATES

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We study a class of inverse problems (IP) of finding kinematical external actions producing the required residual deflections of a physically nonlinear viscoelastic plate over a given time (in a geometrically linear formulation). The correctness of the IP is shown, and the iterative method used in solving the problems is substantiated. The upper bound of residual stresses occurring in the plate after removal of external loads is estimated. Some numerical examples are given.

1. Formulation of the Inverse Problem. We consider a viscoelastic plate of constant thickness h whose median plane Ox_1x_2 occupies region S bounded by a contour γ, the z axis being perpendicular to this plane. We assume that at any time t the deflection $w = w(x_1, x_2, t)$ is negligible as compared with h, and, hence, for total strains we have [1]

$$\varepsilon_{kl} = -zw_{,kl}, \tag{1.1}$$

which corresponds to the state of pure bending. In (1.1) and below, $k, l = 1, 2$; the subscript after the comma denotes the derivative with respect to the corresponding coordinate.

The equations of equilibrium are written as [1]

$$Q_k = M_{kl,l}, \quad Q_{k,k} = -q, \quad Q_k = \int\limits_{-h/2}^{h/2} \sigma_{3k}dz, \quad M_{kl} = \int\limits_{-h/2}^{h/2} \sigma_{kl}zdz. \tag{1.2}$$

Here Q_k and M_{kl} are the cutting forces and moments, q is the intensity of external load, and σ_{kl} are the stress components. Summation from 1 to 2 is performed over repeat subscripts.

From (1.1) and (1.2) follows the equation of virtual works [1]

$$\int\limits_V \sigma_{kl}\varepsilon_{kl}dV = \int\limits_S qwdS + \int\limits_\gamma ((Q + \partial H/\partial s)w - G\partial w/\partial n)ds, \tag{1.3}$$

where $Q = Q_kn_k$, $H = M_{kl}n_kt_l$, $G = M_{kl}n_kn_l$, n_k and t_l are the components of unit normal and tangent vectors to the contour γ, s is the arc length of the contour, and

$$\int\limits_V \cdots dV = \int\limits_{-h/2}^{h/2} \int\limits_S \cdots dSdz.$$

It should be noted that the fields ε_{kl} and σ_{kl} in (1.3) can be unrelated.

The governing relations for the strain of the plate material are written as

$$\varepsilon_{kl} = a_{klmn}\sigma_{mn} + \varepsilon^c_{kl}. \tag{1.4}$$

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Here a_{klmn} and ε^c_{kl} are the components of the elastic compliances and viscous strain (creep strains). We assume that the creep-strain rates $\eta_{kl} = \dot{\varepsilon}^c_{kl}$ depend only on stresses and satisfy the following condition, which generalizes the stability postulate [2]:

$$\Delta\eta_{kl}\Delta\sigma_{kl} \geqslant \lambda a_{klmn}\Delta\sigma_{kl}\Delta\sigma_{mn}, \quad \lambda = \text{const}, \quad \lambda > 0, \quad \Delta\sigma_{kl} = \sigma^{(1)}_{kl} - \sigma^{(2)}_{kl}, \quad \Delta\eta_{kl} = \eta_{kl}(\sigma^{(1)}_{mn}) - \eta_{kl}(\sigma^{(2)}_{mn}). \tag{1.5}$$

Thus, if

$$\eta_{kl} = \Gamma\partial\Sigma/\partial\sigma_{kl}, \quad \Gamma = \Gamma(\Sigma), \quad \Sigma = (a_{klmn}\sigma_{kl}\sigma_{mn})^{1/2}, \tag{1.6}$$

one can say that (1.5) is satisfied with the proviso that $\Gamma' \geqslant \Gamma/\Sigma \geqslant \lambda$, which, in turn, is ensured if $\Gamma(0) = 0$, $\Gamma'(0) = \lambda > 0$, and $\Gamma''(\Sigma) \geqslant 0$. The latter conditions are satisfied, for example, by the following functions used in creep theory [2]: $\Gamma = B[\exp(\lambda\Sigma/B) - 1]$, $\Gamma = B\sinh(\lambda\Sigma/B)$, and $\Gamma = B\Sigma/(B/\lambda - \Sigma)$ ($B = \text{const}$).

The IP considered below involves searching for the kinematical actions producing the required residual shape of the plate over the given time t_*, i.e., one should select a deformation path $w = w(x_1, x_2, t)$ such that at time $t = t_*$ the deflection w takes the given value $\tilde{w}_*(x_1, x_2)$ under zero external load q. It is obvious that the above-mentioned path is not unique. Therefore, we distinguish a class of actions such that the deflection w varies in time according to the given law but with an unknown value w_* for $t = t_*$, after which the corresponding load $q_* = q(x_1, x_2, t_*)$ is instantaneously removed, so that after elastic unbending the residual deflection is $\tilde{w}(x_1, x_2, t_*) = \tilde{w}_*(x_1, x_2)$.

Thus, we consider the following IP class: it is required to determine a function $w_* = w_*(x_1, x_2)$ such that at the current deflection $w(x_1, x_2, t) = f(t)w_*(x_1, x_2)$ $(0 \leqslant t \leqslant t_*)$ [$f(t)$ is the given function, $f(0) = 0$ and $f(t_*) = 1$)] at $t = t_*$, after instantaneous removal of the external load $q_* = q(x_1, x_2, t_*)$ and elastic unloading, the residual deflection $\tilde{w}(x_1, x_2, t_*)$ takes the given value $\tilde{w}_*(x_1, x_2)$.

In this case it is assumed that for $t < 0$ the plate is in an undeformed state and, hence, $\varepsilon^c_{kl} = 0$ for $t = 0$ everywhere in the plate.

We give several remarks on the process of instantaneous unloading at $t = t_*$ and formulate the necessary boundary conditions for γ. The deflection w_* is representable in the form $w_* = w^e_* + \tilde{w}_*$, where $\tilde{w}_*$ is the given residual deflection and w^e_* is a value of elastic unbending that is a solution of the pure elastic problem for the external load $q_* = q(x_1, x_2, t_*)$ subject to appropriate boundary conditions. In this case, for stresses at $t = t_*$ we have [1, 2]

$$\sigma_{kl*} = \sigma^e_{kl*} + \rho_{kl*}, \tag{1.7}$$

where ρ_{kl*} are the residual stresses occurring after elastic unloading and σ^e_{kl*} are the elastic-stress components that correspond to the deflection w^e_*, i.e.,

$$\sigma^e_{kl*} = b_{klmn}\varepsilon^e_{mn*} = -zb_{klmn}w^e_{*,mn}. \tag{1.8}$$

Here b_{klmn} are the components of the elastic-modulus tensor, which is inverse to a_{klmn}, so that

$$a_{klmn}b_{klij} = \delta_{im}\delta_{jn} \tag{1.9}$$

(δ_{im} are the components of the unit tensor).

Given the external load $q_*(x_1, x_2)$, the equation for w^e_*, in view of (1.2) and (1.8), is of the form $b_{klmn}w^e_{*,klmn} = 12\,h^{-3}q_*$.

In turn, the quantity q_* is related to the moments M_{kl*}, which correspond to the stresses σ_{kl*}, by the relation resulting from (1.2): $q_* = -M_{kl*,kl}$. From the last two equalities we obtain

$$b_{klmn}w^e_{*,klmn} = -12\,h^{-3}M_{kl*,kl}. \tag{1.10}$$

As the boundary conditions on γ for unloading at $t = t_*$, one can use one of the following relations [1, 2]:

$$w^e_* = \partial w^e_*/\partial n = 0; \tag{1.11a}$$

$$w^e_* = \tilde{G}_* = 0; \tag{1.11b}$$

$$\tilde{G}_* = \tilde{Q}_* + \partial\tilde{H}_*/\partial s = 0; \tag{1.11c}$$

$$\partial w_*^e/\partial n = \tilde{Q}_* + \partial \tilde{H}_*/\partial s = 0 \tag{1.11d}$$

(the tilde denotes the quantities characterizing the force effect after unloading). Conditions (1.11a)–(1.11c) imply, respectively, that in unloading the contour γ is clamped, free-supported, and free of loads.

One can easily see that, given the function $w_*(x_1, x_2)$, the components $\sigma_{kl*}(x_1, x_2, z) = \sigma_{kl}(x_1, x_2, z, t_*)$ are determined uniquely from the system [which follows from (1.1) and (1.4)]

$$a_{klmn}\dot{\sigma}_{mn} + \eta_{kl}(\sigma_{mn}) = -z\dot{f}w_{*,kl} \tag{1.12}$$

subject to the initial conditions $\sigma_{kl}(x_1, x_2, z, 0) = 0$, since from (1.1) and (1.4) for $t = 0$ we have $a_{klmn}\sigma_{mn} = 0$, because $\varepsilon_{kl}^e = 0$ and $w = 0$.

Thus, the right-hand side of Eq. (1.10) is defined by the function $w_* = w_*(x_1, x_2)$, and hence, the solution of the boundary-value problem (1.10) and (1.11) depends on w_*. Some properties of the operator $w_*^e = w_*^e(w_*)$ are established below.

If we denote the right-hand side of (1.10) by $\Psi(w_*)$, taking into account the equality $w_*^e = w_* - \tilde{w}_*$, for the unknown function $w_* = w_*(x_1, x_2)$, from (1.10) we obtain the equation $b_{klmn}w_{*,klmn} - \Psi(w_*) = b_{klmn}\tilde{w}_{*,klmn}$; in the general case of nonlinear dependences $\eta_{kl} = \eta_{kl}(\sigma_{mn})$, the operator $\Psi = \Psi(w_*)$ cannot be written in explicit form. Nevertheless, under certain simple constraints on the function $f = f(t)$, the IP considered can be reduced to a sequence of direct problems of the type of (1.10) with the known right-hand side and boundary conditions (1.11), whose solutions reduce to the solution of the IP.

2. Correctness of the Inverse Problem. Let us determine the conditions of existence of a single generalized solution of the formulated IP.

We introduce the notation

$$I_1(\sigma_{kl}) = \Big(\int\limits_V \frac{1}{2} a_{klmn}\sigma_{kl}\sigma_{mn}dV\Big)^{1/2}, \quad I_2(\varepsilon_{kl}) = \Big(\int\limits_V \frac{1}{2} b_{klmn}\varepsilon_{kl}\varepsilon_{mn}dV\Big)^{1/2}.$$

Let the deflection field $w = w(x_1, x_2)$ be given. Denote by $\bar{\sigma}_{kl}^e$ elastic stresses that correspond to these deflections, i.e., $\bar{\sigma}_{kl}^e = b_{klmn}\bar{\varepsilon}_{mn}^e = -zb_{klmn}w_{,mn}$. Let

$$\|w\| = I_1(\bar{\sigma}_{kl}^e) = I_2(\bar{\varepsilon}_{kl}^e) = \Big(\int\limits_S (h^3/24)b_{klmn}w_{,kl}w_{,mn}dS\Big)^{1/2}.$$

As is known from [3], if at three points (that do not lie on the same straight line) on a plate the deflection $w = 0$ [for example, $w = 0$ on the contour γ, as is the case with boundary conditions of the form of (1.1a) or (1.12b)], then the quantity $\|w\|$ is a norm which is equivalent to $\|w\|_{H^2(S)}$, the space $H^2(S)$ being full with respect to the introduced norm. The latter is generated by the scalar product

$$(w_1, w_2) = \int\limits_S (h^3/24)b_{klmn}w_{,kl}^{(1)}w_{,mn}^{(2)}dS.$$

In the general case, the quantity $\|w\|$ is a seminorm, and from the equality $\|w\| = 0$ it follows that w is a linear function of x_1 and x_2.

Let the kinematical actions be given by $w^{(i)} = f(t)w_*^{(i)}$ ($i = 1,\ 2$), where $f(t)$ ($0 \leqslant t \leqslant t_*$) is a known function; at the moment of unloading at $t = t_*$, one of the boundary conditions (1.11) is satisfied simultaneously for both actions. In this case, it is assumed that the deflections $w_*^{(i)} = w_*^{(i)}(x_1, x_2)$ and the associated stresses $\sigma_{kl*}^{(i)}$ determined by (1.12), and, hence, the moments $M_{kl*}^{(i)}$ exhibit the required smoothness, so that conditions (1.11) make sense.

As before, the differences of the corresponding quantities are denoted by the symbol Δ. We estimate $\|\Delta w_*^e\|$ in terms of $\|\Delta w_*\|$. Since $I_1(\Delta\sigma_{kl*}^e) = \|\Delta w_*^e\|$, from (1.7) and the equality [1, 2]

$$\int\limits_V a_{klmn}\Delta\sigma_{mn*}^e\Delta\rho_{kl*}dV = 0,$$

which results from (1.3) and (1.11), we obtain

$$I_1^2(\Delta\sigma_{kl*}) = \|\Delta w_*^e\|^2 + I_1^2(\Delta\rho_{kl*}) \geqslant \|\Delta w_*^e\|^2. \tag{2.1}$$

At any moment t $(0 \leqslant t \leqslant t_*)$, in view of (1.4) and (1.9), we have

$$\frac{1}{2}\int\limits_V (a_{klmn}\Delta\dot{\sigma}_{mn}\Delta\sigma_{kl} + \Delta\eta_{kl}\Delta\sigma_{kl})dV = \frac{1}{2}\int\limits_V \Delta\dot{\varepsilon}_{kl}\Delta\sigma_{kl}dV = \frac{1}{2}\int\limits_V a_{klmn}\Delta\sigma_{mn}(b_{klij}\Delta\dot{\varepsilon}_{ij})dV$$
$$\leqslant I_1(\Delta\sigma_{kl})I_1(b_{klmn}\Delta\dot{\varepsilon}_{mn}) = I_1(\Delta\sigma_{kl})I_2(\Delta\dot{\varepsilon}_{kl}), \tag{2.2}$$

where the known inequality

$$\frac{1}{2}\int\limits_V a_{klmn}x_{kl}y_{mn}dV \leqslant I_1(x_{kl})I_1(y_{kl})$$

for $x_{kl} = \Delta\sigma_{kl}$ and $y_{kl} = b_{klmn}\Delta\dot{\varepsilon}_{mn}$ was used.

Taking into account inequality (1.5) and the equality

$$\frac{1}{2}\int\limits_V a_{klmn}\Delta\dot{\sigma}_{mn}\Delta\sigma_{kl}dV = \dot{I}_1(\Delta\sigma_{kl})I_1(\Delta\sigma_{kl}), \quad I_2(\Delta\dot{\varepsilon}_{kl}) = \|\Delta\dot{w}\| = |\dot{f}|\,\|\Delta w_*\|,$$

from (2.2) we find

$$\dot{I}_1(\Delta\sigma_{kl}) + \lambda I_1(\Delta\sigma_{kl}) \leqslant |\dot{f}|\|\Delta w_*\| \quad \text{or} \quad \frac{d}{dt}[I_1(\Delta\sigma_{kl})\exp(\lambda t)] \leqslant \|\Delta w_*\|\,|\dot{f}|\exp(\lambda t).$$

Integrating this inequality with respect to time from zero to t_* and taking into account that $\Delta\sigma_{kl} = 0$ for $t = 0$ everywhere in the plate [since $\Delta w = 0$ at $t = 0$, because $f(0) = 0$], we obtain

$$I_1(\Delta\sigma_{kl*}) \leqslant \beta\|\Delta w_*\|, \quad \beta = \exp(-\lambda t_*)\int\limits_0^{t_*}|\dot{f}|\exp(\lambda t)dt. \tag{2.3}$$

From (2.1) and (2.3) follows the required estimate

$$\|\Delta w_*^e\| \leqslant \beta\|\Delta w_*\|. \tag{2.4}$$

Since $\|\Delta w_*\| = \|\Delta w_*^e + \Delta\tilde{w}_*\| \leqslant \|\Delta w_*^e\| + \|\Delta\tilde{w}_*\|$, it follows from (2.4) that $(1-\beta)\|\Delta w_*\| \leqslant \|\Delta\tilde{w}_*\|$, which, for $\beta < 1$, guarantees uniqueness of the solution of the problem considered and continuity of the operator $w_* = w_*(\tilde{w}_*)$.

It should be noted that the inequality $\beta < 1$ takes place for any function $f = f(t)$ that increases monotonically from zero to unity. Indeed, in this case, $|\dot{f}| = \dot{f}$, and from (2.3) we find

$$\beta = 1 - \lambda\exp(-\lambda t_*)\int\limits_0^{t_*} f\exp(\lambda t)dt < 1,$$

whence, in particular, it follows that the minimum value of β corresponds to the relaxation regime of deformation in which $f(0) = 0$, $\dot{f} > 0$ $(0 < t < t_0)$, and $f = 1$ $(t_0 \leqslant t \leqslant t_*)$ as $t_0 \to 0$. One can easily see that $\beta_{\min} = \exp(-\lambda t_*)$, i.e., in this case

$$\int\limits_0^{t_*}|\dot{f}|\exp(\lambda t)dt = 1.$$

For any other regime,

$$\int\limits_0^{t_*}|\dot{f}|\exp(\lambda t)dt \geqslant \int\limits_0^{t_*}|\dot{f}|dt \geqslant \int\limits_0^{t_*}\dot{f}dt = 1.$$

To prove the existence of a generalized solution, we note that this problem reduces to solution of the following functional equation in the region S:

$$w_* = F(w_*), \qquad F(w_*) = w_*^e(w_*) + \tilde{w}_*. \tag{2.5}$$

Here $\tilde{w}_* = \tilde{w}_*(x_1, x_2)$ is a prescribed function and the operator $w_*^e = w_*^e(w_*)$ was defined in Sec. 1.

Let us first consider the case in which the boundary conditions are of the form (1.11a). Then one can show that if $w_*^{(i)} \in H^2(S)$ $(i = 1, 2)$ the estimate (2.4) holds.

Let $\tilde{w}_* \in H^2(S)$, i.e., $\|\tilde{w}_*\| < \infty$. Then the operator F from (2.5) transforms each element $w_* \in H^2(S)$ into an element of the same space, since

$$\|F(w_*)\| = \|w_*^e + \tilde{w}_*\| \leqslant \|w_*^e\| + \|\tilde{w}_*\| \leqslant \beta\|w_*\| + \|\tilde{w}_*\| < \infty$$

[here we used inequality (2.4) for $w_*^{e(1)} = w_*^e$, $w_*^{e(2)} = 0$, $w_*^{(1)} = w_*$, and $w_*^{(2)} = 0$]. Moreover, for $\beta < 1$ the operator is a contraction operator, since

$$\|F(w_*^{(1)}) - F(w_*^{(2)})\| = \|w_*^e(w_*^{(1)}) - w_*^e(w_*^{(2)})\| \leqslant \beta\|w_*^{(1)} - w_*^{(2)}\|$$

in view of (2.4). From the principle of contractive mappings [4], it follows that there is a unique solution $w_* \in H^2(S)$ of Eq. (2.5), and it can be obtained as the limit of the sequence $\{w_*^n\}$, where $w_*^{n+1} = F(w_*^n)$, i.e., according to (2.5),

$$w_*^{n+1} = w_*^e(w_*^n) + \tilde{w}_* \quad (n = 0, 1, 2, \ldots), \tag{2.6}$$

and w_*^0 is an arbitrary element from $H^2(S)$.

As to the other boundary conditions, we note that in order that (1.11b) or (1.11c) and (1.11d) make sense, it is sufficient to assume that $\tilde{w}_* \in H^3(S)$ [accordingly, $\tilde{w}_* \in H^4(S)$]. Then, with certain smoothness of the function $\eta_{kl} = \eta_{kl}(\sigma_{mn})$ from (1.12), the operator F from (2.5) will transform any element $w_* \in H^3(S)$ [or $w_* \in H^4(S)$] into an element of the same space, since one can show that $w_*^e \in H^3(S)$ [or $w_*^e \in H^4(S)$]. Therefore, in both cases, all elements of sequence (2.6) belong to the corresponding space [$(H^3(S)$ or $H^4(S)$] if w_*^0 belongs to this space. Since the operator F is a contraction operator in the space $H^2(S)$, the limit of sequence (2.6) for $n \to \infty$ will belong, at least, to this space, i.e., $w_* \in H^2(S) \supset H^3(S) \supset H^4(S)$. As one can easily see, in the case of boundary conditions (1.11c) and (1.11d), the deflection w_* is determined with accuracy to an arbitrary linear function of x_1 and x_2 and to an arbitrary constant, respectively.

As was mentioned in Sec. 1, given the function $w_*^n = w_*^n(x_1, x_2)$, the deflection $w_*^{en} = w_*^e(w_*^n)$ is a solution of boundary-value problem (1.10) and (1.11). Therefore, in each iteration, the IP reduces to the direct problem for w_*^e. The rate of convergence of successive approximations to the exact solution is determined by the known inequality [4]

$$\|w_*^n - w_*\| \leqslant \beta^n\|w_*^1 - w_*^0\|/(1 - \beta) \qquad (0 < \beta < 1).$$

It should be noted that the uniqueness of solution of the IP can be proved under weaker conditions than those used above, in particular, in place of (1.5), it is sufficient to postulate the validity of the inequality [2]

$$I_3(t) \equiv \int_0^t \int_V \Delta\eta_{kl}\Delta\sigma_{kl}dVdt \geqslant 0,$$

where the equality sign is possible only for $\Delta\sigma_{kl}(\tau) = 0$ $(0 \leqslant \tau < t)$ everywhere in V.

Indeed, integrating inequality (2.2) with respect to time from zero to the current time t, we obtain

$$I_1^2(\Delta\sigma_{kl}(t)) + I_3(t) \leqslant 2\int_0^t I_1(\Delta\sigma_{kl}(t))\|\Delta\dot{w}(t)\|dt \equiv 2\Phi(t). \tag{2.7}$$

Since $I_3(t) \geqslant 0$ and $I_1(\Delta\sigma_{kl}) = \dot{\Phi}/\|\Delta\dot{w}\|$, from (2.7) we have $(\dot{\Phi})^2 \leqslant 2\Phi\|\Delta\dot{w}\|^2$ or $\dot{\Phi}\Phi^{-1/2} \leqslant \sqrt{2}\|\Delta\dot{w}\|$.

Integrating the latter inequality with respect to time and taking into account that $\Phi(0) = 0$, we find that

$$2\Phi^{1/2} \leqslant \sqrt{2}\int\limits_0^t \|\Delta\dot{w}\|dt,$$

and, hence,

$$2\Phi \leqslant \Big(\int\limits_0^t \|\Delta\dot{w}\|dt\Big)^2 = \|\Delta w_*\|^2\Big(\int\limits_0^t |\dot{f}|dt\Big)^2. \tag{2.8}$$

Since $\Delta\tilde{w}_* = 0$, we have $\|\Delta w_*\| = \|\Delta w_*^e\|$, and taking into account (2.1), from (2.7) and (2.8) we obtain

$$I_1^2(\Delta\rho_{kl*}) + I_3(t_*) \leqslant (\beta_0^2 - 1)\|\Delta w_*^e\|^2, \quad \beta_0 = \int\limits_0^{t_*} |\dot{f}|dt,$$

which, for $\dot{f} \geqslant 0$ and $\beta_0 = 1$, is possible only if $\Delta\sigma_{kl}(\tau) = 0$ $(0 \leqslant \tau < t_*)$, whence, in turn, follows uniqueness of the solution of this problem in the above-mentioned sense.

3. Estimate of the Level of Residual Stresses for $t = t_*$. Inequalities of the type of (2.3) and (2.4) yield an upper bound for the level of residual stresses arising in the plate at $t = t_*$ after elastic unloading. As a measure that characterizes this level, we choose $I_* \equiv I_1^2(\rho_{kl*})$.

It should be noted that the formulas obtained in Sec. 2 for the differences between the corresponding quantities are also valid for the quantities themselves, i.e., the sign Δ can be omitted. This follows from the fact that as the first, one can take the main stress-strain state, and as the second, the natural state corresponding to zero deflections, deformations, and stresses everywhere in the plate.

Bearing in mind this remark, from (2.1) and (2.3) we obtain

$$I_* \leqslant \beta^2\|w_*\|^2 - \|w_*^e\|^2 \quad (\beta < 1). \tag{3.1}$$

Since, according to (2.4),

$$\|w_*^e\| \leqslant \beta\|w_*\| = \beta\|w_*^e + \tilde{w}_*\| \leqslant \beta(\|w_*^e\| + \|\tilde{w}_*\|)$$

i.e., $\|w_*^e\| \leqslant \beta(1-\beta)^{-1}\|\tilde{w}_*\|$, and

$$\|w_*\|^2 \leqslant \|w_*^e\|^2 + 2\|w_*^e\|\,\|\tilde{w}_*\| + \|\tilde{w}_*\|^2 \leqslant \|w_*^e\|^2 + [2\beta(1-\beta)^{-1} + 1]\|\tilde{w}_*\|^2,$$

from (3.1) we find

$$I_* \leqslant (\beta^2 - 1)\|w_*^e\|^2 + \beta^2(1+\beta)(1-\beta)^{-1}\|\tilde{w}_*\|^2 \leqslant \beta^2(1+\beta)(1-\beta)^{-1}\|\tilde{w}_*\|^2.$$

Hence, it is evident that the minimum estimate for I_* is obtained for minimum β. As was noted above, $\beta_{\min} = \exp(-\lambda t_*)$, which corresponds to the relaxation regime of deformation.

4. Numerical Examples. We consider a square plate of thickness $h = 6$ whose middle plane occupies area S: $0 \leqslant x_i \leqslant 300$ $(i = 1, 2)$ (hereafter all dimensions are given in millimeters). The residual deflection for $t = t_*$ is given by $\tilde{w}_* = -9\cdot 10^{-4}[x_1(x_1 - 300) + x_2(x_2 - 300)]$.

The plate material is considered isotropic, and the governing strain relations for the plate are of the form of (1.4), where the elastic strains ε_{kl}^e and the viscous-strain rates η_{kl} are given by

$$\varepsilon_{kl}^e = (3/2)\sigma_{kl}^0/E \qquad (E \text{ is Young's modulus}); \tag{4.1}$$

$$\eta_{kl} = (3/2)A[\exp(\alpha\sigma_i) - 1]\sigma_{kl}^0/\sigma_i \qquad (A \text{ and } \alpha \text{ are constants}). \tag{4.2}$$

In formulas (4.1) and (4.2), $\sigma_{kl}^0 = \sigma_{kl} - (1/3)\sigma_{nn}\delta_{kl}$ are the stress-deviator components and $\sigma_i = (\sigma_{11}^2 + \sigma_{22}^2 - \sigma_{11}\sigma_{22} + 3\sigma_{12}^2)^{1/2}$ is the stress intensity.

One can easily see that relations (4.2) are a particular case of relations (1.6) which contain the function $\Gamma = B[\exp(\lambda\Sigma/B) - 1]$ mentioned in Sec. 1. Here $\Sigma = \sigma_i/\sqrt{E}$, $B = A\sqrt{E}$, and $\lambda = \alpha AE$. In this case, inequality (1.5) is valid.

As the function $f = f(t)$ occurring in the condition of the problem we use

$$f(t) = C[(1 - t/t_*) - (1 - t/t_*)^{æ}] + t/t_*, \tag{4.3}$$

where $C \geqslant 0$ and $æ > 0$ are constants.

As was mentioned above, for the constant β from (2.3) the inequality $\beta < 1$ (which guarantees the correctness of the problem) is satisfied for $\dot{f} \geqslant 0$. In the case considered, in order that the condition $\dot{f} = [1 - C + Cæ(1 - t/t_*)^{æ-1}]/t_* \geqslant 0$ be satisfied, it is sufficient that

(a) $1 - C \geqslant 0$, $C \geqslant 0$, and $æ > 0$, i.e., $0 \leqslant C \leqslant 1$ and $æ > 0$,

or

(b) $\dot{f}(0) \geqslant 0$ and $\ddot{f} = -Cæ(æ - 1)(1 - t/t_*)^{æ-2}/t_*^2 \geqslant 0$, i.e., $C > 0$ and $1 - 1/C \leqslant æ \leqslant 1$.

If $æ > 1$ and $C > 0$, then $|\dot{f}| \leqslant [|1 - C| + Cæ(1 - t/t_*)^{æ-1}]/t_*$, and, hence, integrating by parts, we obtain

$$\int_0^{t_*} |\dot{f}| \exp(\lambda t) dt \leqslant |1 - C|[\exp(\lambda t_*) - 1]/(\lambda t_*) + C + C\lambda \int_0^{t_*} (1 - t/t_*)^{æ} \exp(\lambda t) dt$$

$$< |1 - C|[\exp(\lambda t_*) - 1]/(\lambda t_*) + C + C\lambda \int_0^{t_*} (1 - t/t_*) \exp(\lambda t) dt = (|1 - C| + C)[\exp(\lambda t_*) - 1]/(\lambda t_*).$$

Consequently, in this case, the condition $\beta < 1$ is satisfied if

(c) $(|1 - C| + C)[1 - \exp(-\lambda t_*)]/(\lambda t_*) \leqslant 1$.

The numerical solution of this problem is based on iterative process (2.6). Let the nth approximation for the unknown function w_*^n be known. The procedure of finding the $(n+1)$th approximation for w_*^{n+1} is as follows. From (1.1), (1.4), (4.1), and (4.2) we find a system of equations of the form of (1.12),

$$(3/2)\dot{\sigma}_{kl}^{n0}/E + (3/2)A[\exp(\alpha\sigma_i^n) - 1]\sigma_{kl}^{n0}/\sigma_i^n = -z\dot{f}w_{*,kl}^n, \tag{4.4}$$

subject to the initial conditions $\sigma_{kl}^n = 0$ for $t = 0$. Integrating the system, we determine the stress components σ_{kl*}^n of the nth approximation prior to unloading for $t = t_*$ and the corresponding moments M_{kl*}^n.

Then, to find the elastic deflection w_*^{en}, we obtain an equation of the form of (1.10),

$$D\Delta\Delta w_*^{en} = -M_{kl*,kl}^n, \tag{4.5}$$

which in this case is biharmonic [2, 5] and is subject to one of the boundary conditions (1.11) ($D = Eh^3/9$ is the cylindrical stiffness of the plate).

From the known function w_*^{en}, according to (2.6), we find the $(n+1)$th approximation for $w_*^{n+1} = w_*^{en} + \tilde{w}_*$. The procedure is then repeated. As a zeroth approximation for the desired function w_* we use $w_*^0 = \tilde{w}_*$.

The following values of the constants were used in the calculations: $E = 66{,}700$ MPa, $A = 2.008 \cdot 10^{-9}$ sec^{-1}, $\alpha = 0.13$ MPa^{-1}, and $t_* = 10$ h. It was assumed that at the moment of unloading ($t = t_*$) the contour γ is absolutely free, and this corresponds to boundary condition (1.11c). In each iteration problem (4.5) and (1.11c) was solved by the finite-element method. For this, a triangular element with a cubic dependence of the form function on the uniform L coordinates of the triangle was used [6]. In view of symmetry, only a quarter of the plate was considered ($0 \leqslant x_i \leqslant 150$, $i = 1, 2$). The quarter was divided into 18 triangles (16 nodes). In calculating the moments M_{kl*}, we used the Simpson formula with 13 integration nodes along the plate thickness. System (4.4) was solved at each separation node in each iteration (in the plane Ox_1x_2 and along the z coordinate) using the Runge–Kutta–Merson method with automatic selection of time steps. Iteration process (2.6) was terminated when the condition $\max_{x_1,x_2} |1 - \tilde{w}_*^n(x_1, x_2)/\tilde{w}_*(x_1, x_2)| \leqslant 10^{-3}$ was satisfied.

Calculations were performed for various values of the parameters C and $æ$ from (4.3). A proper choice of the parameters allows one to describe different deformation regimes, for example:

(1) a monotonic increase in deflection from 0 to w_* with time ($\dot{f} \geqslant 0$, and the appropriate restrictions on C and $æ$ were given above);

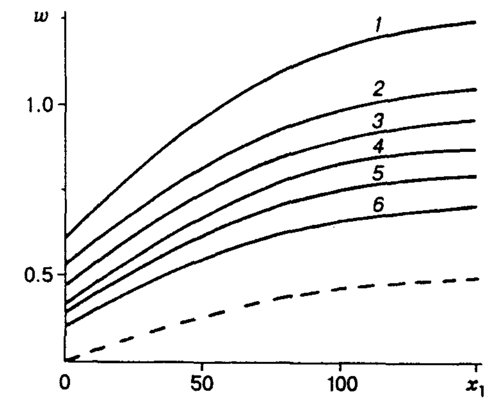

Fig. 1

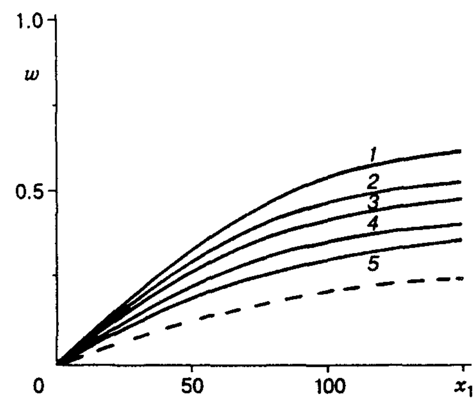

Fig. 2

(2) deformation close to relaxation [1, 2, 5], i.e., an almost instantaneous growth in deflection to w_* with subsequent fixation up to time $t = t_*$ ($C = 1$ and $æ \gg 1$);

(3) a monotonic increase in deflection to a value greater than w_* with a subsequent monotonic decrease to w_* ($C > 1$ and $æ > 1$).

Figures 1 and 2 show the graphs for the given residual deflection $\tilde{w}_*$ (dashed curves) and the desired deflection w_* (solid curves) in the cross sections $x_2 = 150$ and 0 for $C = 1.0, 0.8, 1.3, 1.0, 1.0$, and 1.3, $æ = 1$, 2, 2, 5, 30, and 30 (Fig. 1, curves 1–6), and $C = 1.0, 0.8, 1.3, 1.0$, and 1.3, $æ = 1$, 2, 30, and 30 (Fig. 2, curves 1–5). All these values satisfy the above-mentioned restrictions (a), (b), or (c), which are sufficient for satisfaction of the inequality $\beta < 1$, which guarantees the correctness of the problem considered.

Among the functions of the form of (4.3) determined by the parameters C and æ, it is of interest to find a function for which the level of residual stresses in the plate is minimum after unloading at $t = t_*$. The components ρ_{kl*} are found from (1.7): $\rho_{kl*} = \sigma_{kl*} - \sigma^e_{kl*}$. Here σ^e_{kl*} are related to the deflection w^e_* by the relations of the form (1.8): $\sigma^e_{kl*} = -(2/3)Ez(w^e_{*,kl} + w^e_{*,nn}\delta_{kl})$. As a quantity that characterizes the level of residual stresses, we choose their intensity $\rho_{i*} = (\rho^2_{11*} + \rho^2_{22*} - \rho_{11*}\rho_{22*} + 3\,\rho^2_{12*})^{1/2}$.

For the deformation regimes corresponding to curves 1–6 in Fig. 1, we obtained the following values: $\rho_{i*\max} = 1.09, 0.93, 0.88, 0.77, 0.67$, and 0.69 MPa. Hence one can see that $\min \rho_{i*\max} = 0.67$ MPa corresponds to $C = 1$ and $æ = 30$. A comparative analysis of the results shows that for $C = 1$ and $æ \geqslant 30$ the stress-strain state of the plate (including the diagrams of residual stresses) almost coincides with that for the relaxation regime of deformation. Thus, the latter regime is optimal in terms of the level of residual stresses after unloading. This is in qualitative agreement with the estimate obtained in Sec. 3.

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